# Level-spacing distribution of a fractal matrix

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We diagonalize numerically a Fibonacci matrix with fractal Hilbert space structure of dimension $d_f = 1.8316...$ We show that the density of states is logarithmically normal while the corresponding level-statistics can be described as critical since the nearest-neighbor distribution function approaches the intermediate semi-Poisson curve. We find that the eigenvector amplitudes of this matrix are also critical lying between extended and localized.

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The numerical diagonalization of one-electron Hamiltonians in discrete space tight-binding lattices has been proved a very useful tool to treat the effect of disorder on the motion of quantum particles, alternatively to field theoretic methods. In particular, the direct computation of eigenvalues and eigenvectors combined with finite-size scaling techniques can answer questions about Anderson localization due to disorder [1], quantum chaos in the energy levels or wave functions [2,3], etc. The difficulties become immense when one wants to treat many-body problems in the same framework since the dimension of the Hilbert space increases dramatically (exponentially) with the number of electrons. However, also in this case the nearest-neighbor hopping bounded kinetic energy and the two-body character of the interaction guarantee that the Hamiltonian matrix structure is very sparse, often described as multifractal in the adopted Hilbert space. For example, in ref. [4] multifractal exponents $D_q$ were computed to characterize the Hilbert space structure of interacting many-body Hamiltonians.

We shall treat a much simpler, albeit interesting problem, with a given Hamiltonian which consists of zeroes and ones, neither periodic nor random but with a simple fractal structure. We construct and diagonalize a Fibonacci matrix of order $n$ of size $N_n \times N_n$, with $N_n = N_{n-1} + N_{n-2}$, $N_0 = N_{-1} = 1$. This matrix represents a self-similar fractal object itself and was introduced in ref. [5] in order to obtain the ground state of the square Ising antiferromagnet in a maximum critical field. The Fibonacci matrix is a transfer matrix connecting the possible ground states of the $n \times m$ and $n \times (m+1)$ square lattices. For size $N_n$ it has the block form

$$F_n = \begin{pmatrix} F_{n-1} & G_{n-1} \\ G_{n-1}^T & 0 \end{pmatrix} \tag{1}$$

where $G_{n-1}$ represents the $N_{n-1} \times N_{n-2}$ submatrix of $F_{n-1}$ (its upper left corner), $G_{n-1}^T$ its transpose and 0 denotes the zero $N_{n-2} \times N_{n-2}$ matrix. The sequence of fractal matrices for $n = 1, 2, ...$ begins with the matrix of size $N_1 = 2$

$$F_1 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \tag{2}$$

size $N_2 = 3$

$$F_2 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \tag{3}$$

size $N_3 = 5$

$$F_3 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{pmatrix}, \tag{4}$$

*etc* and the corresponding matrices for higher $n$'s are easily obtained from eq. (1) successively. For example, the matrix $F_{10}$ of size $N_{10} = 144$ is shown in fig. 1.

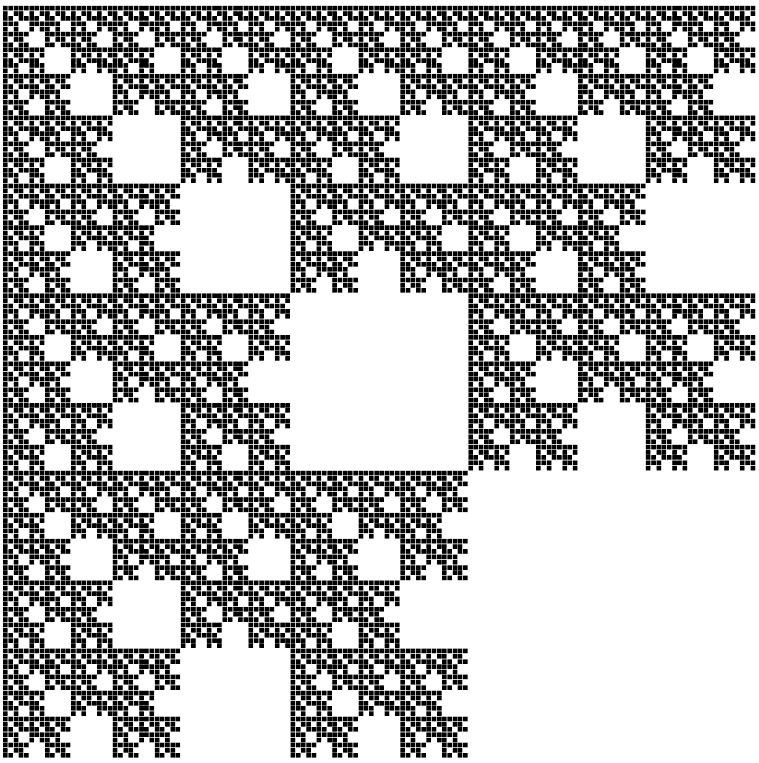

**FIG. 1.** The fractal Fibonacci matrix $F_{10}$ of size $N_{10} = 144$. The ones are denoted with black and the zeroes with white.

In the Fibonacci matrix structure of fig. 1 we observe no coverage of the full two-dimensional Hilbert space, with blocks within blocks, so that the matrix is a simple fractal object. The fractal dimension of the Fibonacci matrix can be obtained form eq. (1) if we associate the mass $M_n$ measured by the number of 1's to the $F_n$ matrix of size $N_n$, via the scaling relation

$$M_n = N_n^{d_f}. \tag{5}$$

The corresponding asymptotic ($n \to \infty$) size ratio $N_{n+1}/N_n = (1+\sqrt{5}))/2$ and mass ratio $M_{n+1}/M_n = 1+\sqrt{2}$ for two successive generations $n$ and $n+1$ can enter eq. (5). The size ratio is obtained from two successive Fibonacci approximants (equals the ratio of the sides of two successive rectangles) and by definition corresponds to the golden mean $(1+\sqrt{5})/2$. The mass ratio $M_{n+1}/M_n = 1+\sqrt{2}$ is obtained from the of the series $M_n = M_{n-1} + 2(M_{n-2} + M_{n-3} + M_{n-4} + ... + M_1 + 2)$, $n = 2, 3, .., M_1 = 3$ and the formation of $M_{n+1}/M_n = 1 + 2(\frac{1}{M_n/M_{n-1}} + \frac{1}{M_n/M_{n-2}} + ... + \frac{1}{M_n/M_1} + \frac{1}{M_n/2})$ which in the limit of $n \to \infty$ where $\frac{M_{n+1}}{M_n} = \frac{M_n}{M_{n-1}} \to x$, $\frac{M_n}{M_{n-2}} = \frac{M_n}{M_{n-1}}\frac{M_{n-1}}{M_{n-2}} \to x^2$, etc, becomes $x = 1 + 2(\frac{1}{x} + \frac{1}{x^2} + ...)$ and by summing the geometric series we obtain $x = 1+\sqrt{2}$. Therefore, the matrix of fig. 1 is a simple fractal (monofractal) object with dimension between 1 and 2 given by

$$d_f = \frac{log(1+\sqrt{2})}{log((1+\sqrt{5})/2)} = 1.8316... \tag{6}$$

In order to obtain the residual entropy of the antiferromagnetic Ising model in maximum critical field the leading eigenvalue of such Fibonacci matrices up to $F_9$ was estimated in ref. [5].

Our purpose is entirely different. We shall consider the full spectrum of the Fibonacci matrix (eigenvalues and eigenvectors) and address questions related to Anderson localization and quantum chaos. We emphasise that we do not have in mind any particular physical system described by this matrix although such fractal matrices might be in general relevant for critical (between extended and localized) one-electron states in disordered or quasiperiodic potentials or possibly many-electrons states for interacting electrons in the presence of disorder [4]. We shall address the following questions concerning the Fibonacci matrix:

(1) What is the density of states?

(2) What is the required 'unfolding' procedure and the nearest-level spacing distribution function?

(3) What is the behavior of the corresponding eigenvectors?

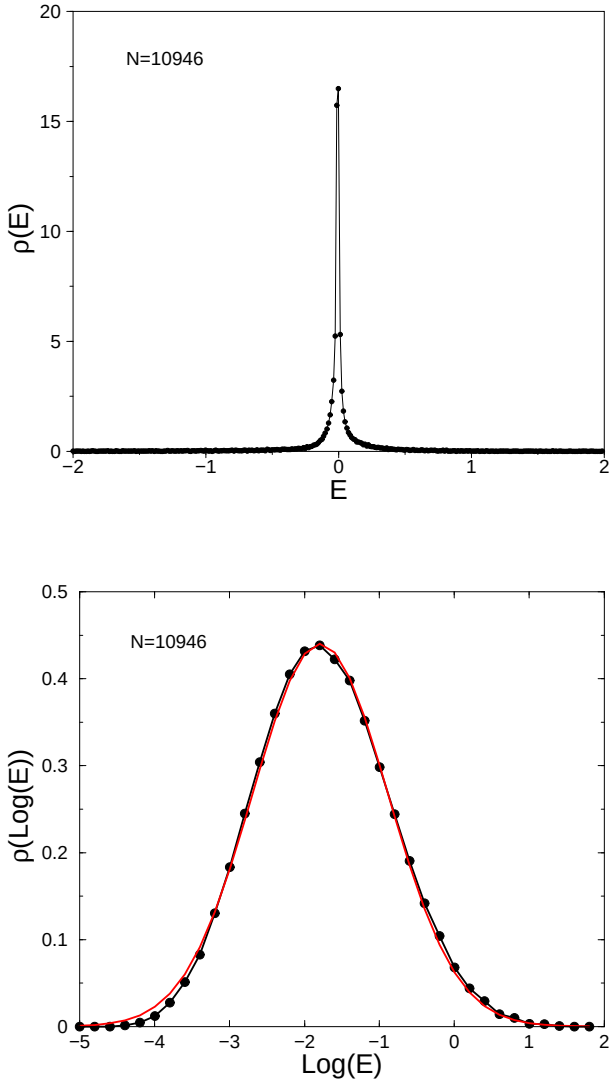


**FIG. 2. (a)** The density of states (full circles joined by a solid line) for the Fibonacci matrix $F_{19}$ of size $N = N_{19} = 10946$. **(b).** The corresponding density of states (full circles joined by a solid line) as a function of $\log(E)$, where $E$ is the absolute value of the energy. The simple Gaussian fit with mean $-1.79$ and standard deviation 0.91 is also shown by the continuous line.

Firstly, we compute the density of states $\rho(E)$ as a function of the energy $E$ which is shown in fig. 2(a). It is seen to be dominated by a strong peak at the origin $E = 0$ and also has very long tails which increase with the size. Instead, the distribution of $\rho(logE)$ shown in fig. 2(b), where $E$ is the absolute value of the energy, leads to a Gaussian-like function. From the Gaussian fit of fig. 2(b) we can conclude that the density of states $\rho(E)$ of the Fibonacci matrix follows a log-normal distribution. We have obtained similar densities of states for all other sizes examined. Having answered the first question, we proceed to compute the energy level-statistics in this system. In order to do so it is required to solve the problem of 'unfolding' for the unusual density of states of fig.2 which demonstrates a strong peak at the origin. It is impossible to obtain a meaningful level-statistics without solving the 'unfolding' problem which is required to make the density of states (particularly this unusually singular density) constant, with mean-level spacing $\Delta = 1$. Our approach consists in 'unfolding' the energy levels by using the density of fig. 2(b) instead of using that of fig. 2(a). In fact, this is essentially a double unfolding procedure of the original spectrum, one going from fig. 2(a) to 2(b) and the second done for the data of fig. 2(b), so finally the mean level-spacing becomes $\Delta = 1$.

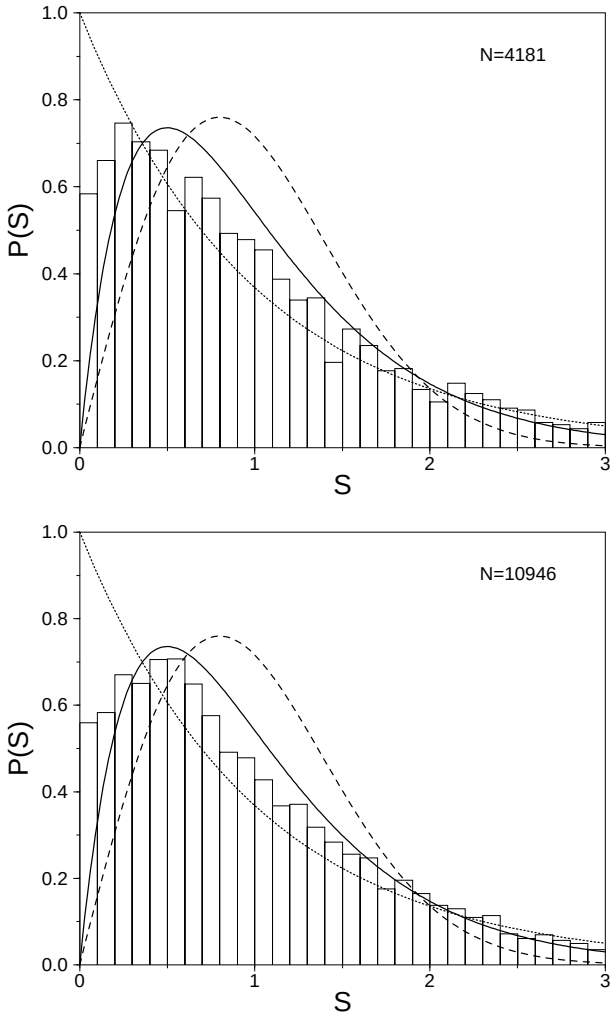


**FIG. 3. (a), (b)** The calculated nearest level-spacing distribution $P(S)$ by unfolding the data for $\rho(log(E))$ for two Fibonacci matrices of sizes $N = N_{17} = 4181, N_{19} = 10946$. The dashed line is the Wigner surmise $P(S) = (\pi/2)Sexp(-(\pi/4)S^2)$ (extended states), the dotted line is the Poisson law $P(S) = exp(-S)$ (localized states) and the solid line is the semi-Poisson distribution $P(S) = 4Sexp(-2S)$ (critical states). The asymptotic distributions for extended and localized states are less relevant for our system since the data seem to approach the semi-Poisson law for the sizes used.

The obtained spacing distribution shown in fig. 3 is neither Wigner (extended) nor Poisson (localized). It is better described by a semi-Poisson intermediate level-statistics known to characterize pseudointegrable billiars [6], disordered systems at the critical point after performing certain average over boundary conditions [7], quasiperiodic critical systems [8,9], interacting particles [4], etc. Although the agreement with the semi-Poisson curve $P(S) = 4Sexp(-2S)$ is not perfect the obtained curve shows some scale-invariant characteristics and we can attribute the small disagreement with the semi-Poisson curve as due to the rather small number of levels considered. For example, the maximum size of 10946 considered in our computations, which gives the same number of levels, is clearly insufficient to answer definitely this issue, since over 100000 levels are usually required for this purpose [7,8], clearly impossible at present. However, we expect the agreement to improve as $N$ increases. The discussion up to now also answers the second question.

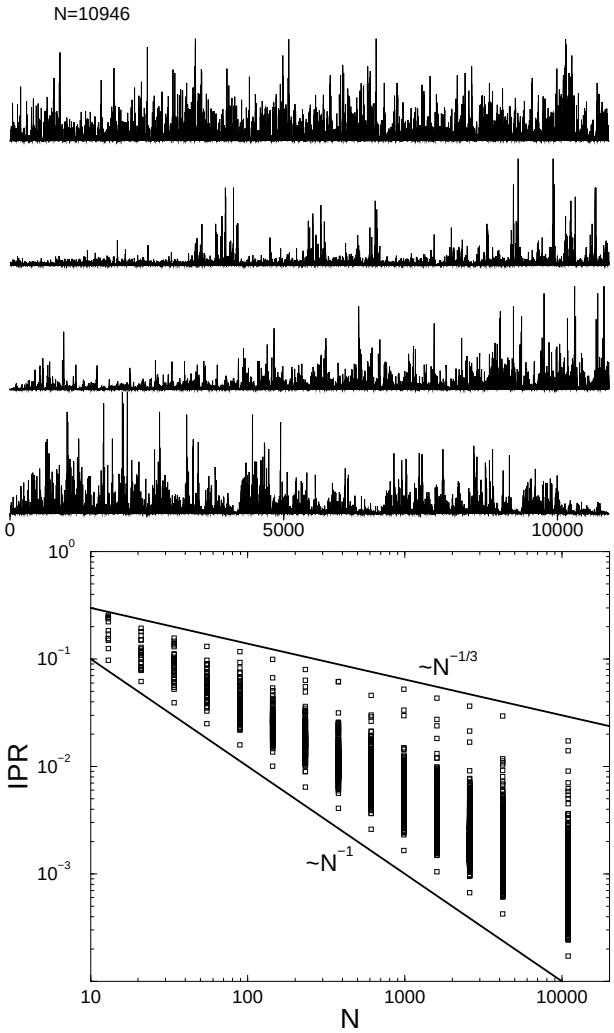


**FIG. 4. (a)** The computed typical eigenvector amplitude distributions for the Fibonacci matrix of size $N = N_{19} = 10946$. **(b)** The corresponding scaling of the inverse participation ratio vs. size $N$ for all eigenstate amplitudes, which defines their $D_2$ fractal exponents to lie between 1/3 and 1.

Finally, in order to answer the third question we have considered the eigenvector behavior of the Fibonacci matrix. Results for some typical eigenvector amplitude distributions together with the scaling of the inverse participation ratio are shown in fig. 4. We find that the vast majority of the eigenvectors are 'critical' sharing delocalized fractal characteristics. The scaling of the inverse participation ratio allows to compute $D_2$ exponent for the eigenvector amplitudes. We obtain $1/3 \leq D_2 \leq 1$ while only very few eigenstates of this matrix are truly extended with $D_2 = 1$ usually lying at the spectral edges while states close to the band center are more 'localized' ($D_2 \approx 1/3$).

In summary, we have obtained numerically the full eigensolutions of a fractal Fibonacci matrix. The results show critical level-statistics with almost semi-Poisson nearest-level distribution and multifractal eigenvectors. The Fibonacci matrix studied shares similar characteristics to one-electron disordered systems at the mobility edge, quasicrystaline and also interacting electron systems. Moreover, it can be interesting for quantum systems which can show intermediate level-statistics of what is believed to be 'critical chaos', in the border between quantum chaos and integrability. In might also offer some hints for the expected behavior from the diagonalization of matrices in complicated many-body Hamiltonians.

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